Sec 2.10: Euler's Numerical Approximation Method

Sometimes getting solution to first order differential equations is very difficult; numerical approximations or geometric schemes can be just as useful.

Euler's Method (Algorithm) Given $y' = f(t, y),\ y(t_0) = y_0$ and t^* in the domain of definition of the solution $y(t)$. Then, we can approximate $y(t^*)$ in n-steps as follows.

1. Compute the **step size** $h = \dfrac{t^* - t_0}{n}$

★ Memorize Equation

2. For $i = 1, 2, \cdots, n$, compute $t_i = t_0 + ih$

$f(t_i) \approx h f(t_{i-1}) + f(t_{i-1})$

3. For $i = 1, 2, \cdots, n$, compute $y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$

4. y_n is an approximation of $y(t^*)$.

Idea of the Proof:

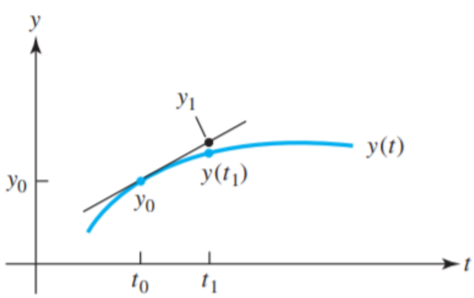


FIGURE 2.19

The line tangent to $y(t)$ at the initial point (t_0, y_0) has slope $f(t_0, y_0)$. Following the tangent line to time t_1, we arrive at the point (t_1, y_1) and have an approximation, y_1, to the solution value, $y(t_1)$.

Ex1. Let $y(t)$ be the solution of the initial value problem $y' = y - t^2$, $y(1) = 2$. Using Euler's method with step size $h = 1/2$ to approximate $y(2)$, one obtains

(a) 2

(b) $\frac{5}{2}$

(c) $\frac{11}{4}$

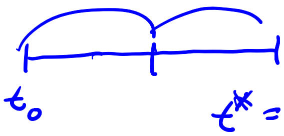
(d) $\frac{21}{8}$

$y' = f(t,y) = y - t^2$

$y(t_0) = 2$

$t_0 = 1$

$h = 1/2$



need to approx. $y(2)$

$$y(t_1) \approx y_1 = y_0 + h\underbrace{f(t_0, y_0)}_{y_0'} = 2 + \frac{1}{2}\left[\underbrace{2}_{y_0} - (\underbrace{1}_{t_0})^2\right] = 2.5$$

$$y(t_2) = y(t^*) = y(2) \approx y_2 = y_1 + h\underbrace{f(t_1, y_1)}_{y_1'} = 2.5 + \frac{1}{2}\left[2.5 - (1.5)^2\right] = 2.625 = \frac{21}{8}$$

Ex2. Given $y' = y + 2$ with $y(0) = 1$, consider $n = 3$ to approximate $y(0.3)$.

$y' = f(t, y) = y + 2$

$y(t_0) = 1$

$t_0 = 0$

$h = \frac{0.3 - 0}{3} = 0.1$

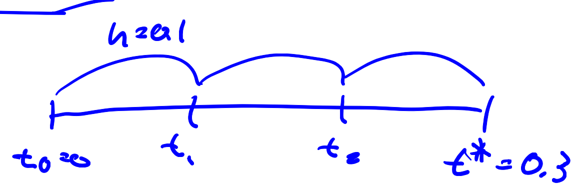
$f(y, t_0)$

My Attempt

$y' = f(t, y) = y + 2$

$y(0) = 1$

$y_0 = y(t_0)$



$$y(t_1) = y(0.1) \approx y_1 = y_0 + 0.1\underbrace{f(t, y_0)}_{y_0'} = 1 + 0.1[1 + 2] = 1 + 0.3 = 1.3$$

$$y(t_2) = y(0.2) \approx y_2 = y_1 + 0.1 f(t, y_1) = 1.3 + 0.1[1.3 + 2] = 1.63$$

$$y(t^*) = y(0.3) \approx y_3 = y_2 + 0.1 f(t, y_2) = 1.63 + 0.1[1.63 + 2] = \boxed{1.993}$$